Electric Potential Function for Charge Densities

Recall the total static electric field produced by 2 **different** charges (or charge densities) is just the **vector sum** of the fields produced by each:

$$\mathbf{E}(\bar{\mathbf{r}}) = \mathbf{E}_1(\bar{\mathbf{r}}) + \mathbf{E}_2(\bar{\mathbf{r}})$$

Since the fields are conservative, we can write this as:

$$\begin{aligned}\mathbf{E}(\bar{\mathbf{r}}) &= \mathbf{E}_1(\bar{\mathbf{r}}) + \mathbf{E}_2(\bar{\mathbf{r}}) \\ -\nabla V(\bar{\mathbf{r}}) &= -\nabla V_1(\bar{\mathbf{r}}) - \nabla V_2(\bar{\mathbf{r}}) \\ -\nabla V(\bar{\mathbf{r}}) &= -\nabla \left(V_1(\bar{\mathbf{r}}) + V_2(\bar{\mathbf{r}})\right)\end{aligned}$$

Therefore, we find,

$$V(\bar{\mathbf{r}}) = V_1(\bar{\mathbf{r}}) + V_2(\bar{\mathbf{r}})$$

In other words, **superposition** also holds for the electric potential function! The total electric potential field produced by a collection of charges is simply the **sum** of the electric potential produced by **each**.

Consider now some **distribution** of charge, $\rho_v(\bar{\mathbf{r}})$. The amount of charge dQ, contained within **small volume** dv, located at position $\bar{\mathbf{r}}'$, is:

$$dQ = \rho_v(\bar{\mathbf{r}}')\, dv'$$

The **electric potential function** produced by this charge is therefore:

$$dV(\bar{\mathbf{r}}) = \frac{dQ}{4\pi\varepsilon_0 \left|\bar{\mathbf{r}}-\bar{\mathbf{r}}'\right|}$$
$$= \frac{\rho_v(\bar{\mathbf{r}}')\,dv'}{4\pi\varepsilon_0 \left|\bar{\mathbf{r}}-\bar{\mathbf{r}}'\right|}$$

Therefore, **integrating** across all the charge in some **volume** V, we get:

$$V(\bar{\mathbf{r}}) = \iiint_V \frac{\rho_v(\bar{\mathbf{r}}')}{4\pi\varepsilon_0 \left|\bar{\mathbf{r}}-\bar{\mathbf{r}}'\right|}\,dv'$$

Likewise, for **surface** or **line** charge density:

$$V(\bar{\mathbf{r}}) = \iint_S \frac{\rho_s(\bar{\mathbf{r}}')}{4\pi\varepsilon_0 \left|\bar{\mathbf{r}}-\bar{\mathbf{r}}'\right|}\,ds'$$

$$V(\bar{\mathbf{r}}) = \int_C \frac{\rho_\ell(\bar{\mathbf{r}}')}{4\pi\varepsilon_0 \left|\bar{\mathbf{r}}-\bar{\mathbf{r}}'\right|}\,d\ell'$$

Note that these integrations are **scalar** integrations—typically they are **easier** to evaluate than the integrations resulting from **Coulomb's Law**.

Once we find the electric potential function $V(\bar{r})$, we can **then** determine the total **electric field** by taking the gradient:

$$\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$$

Thus, we now have **three** (!) potential methods for determining the **electric field** produced by some **charge distribution** $\rho_v(\bar{r})$:

1. Determine $\mathbf{E}(\bar{r})$ from **Coulomb's Law**.

2. If $\rho_v(\bar{r})$ is symmetric, determine $\mathbf{E}(\bar{r})$ from **Gauss's Law**.

3. Determine the **electric potential function** $V(\bar{r})$, and then determine the electric field as $\mathbf{E}(\bar{r}) = -\nabla V(\bar{r})$.

Q: *Yikes! Which of the three should we use??*

A: To a certain extent, it does **not matter**! All three will provide the **same** result (although $\rho_v(\bar{r})$ **must** be symmetric to use method 2!).

However, **if** the charge density is symmetric, we will find that using Gauss's Law (method 2) will **typically** result in much less work!

Otherwise (i.e., for **non**-symmetric $\rho_v(\bar{r})$), we find that **sometimes** method 1 is easiest, but in **other** cases method 3 is a bit less stressful (i.e., **you** decide!).